

# Simulations of Weighted Tree Automata

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**Abstract.** Simulations of weighted tree automata (wta) are considered. It is shown how such simulations can be decomposed into simpler functional and dual functional simulations also called forward and backward simulations. In addition, it is shown in several cases (fields, commutative rings, NOETHERIAN semirings, semiring of natural numbers) that all equivalent wta  $M$  and  $N$  can be joined by a finite chain of simulations. More precisely, in all mentioned cases there exists a single wta that simulates both  $M$  and  $N$ . Those results immediately yield decidability of equivalence provided that the semiring is finitely (and effectively) presented.

## 1 Introduction

Weighted tree automata (or equivalently, weighted tree grammars) are widely used in applications such as model checking [1] and natural language processing [22]. They finitely represent mappings, called tree series, that assign a weight (taken from a semiring) to each tree. For example, a probabilistic parser would return a tree series that assigns to each parse tree its likelihood. Consequently, several toolkits [21, 25, 10] implement weighted tree automata.

The notion of simulation that is used in this paper is a generalization of the simulations for unweighted and weighted (finite) string automata of [5, 15]. The aim is to relate structurally equivalent automata. The results of [5, Section 9.7] and [23] show that two unweighted string automata (i.e., potentially nondeterministic string automata over the BOOLEAN semiring) are equivalent if and only if they can be connected by a finite chain of relational simulations, and that in fact *functional* and *dual functional* simulations are sufficient. Simulations for weighted string automata (wsa) are called *conjugacies* in [2, 3], where it is shown

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that for all fields, many rings including the ring  $\mathbb{Z}$  of integers, and the semiring  $\mathbb{N}$  of natural numbers, two wsa are equivalent if and only if they can be connected by a finite chain of simulations. It is also shown that even a finite chain of functional (*covering*) and dual functional (*co-covering*) simulations is sufficient. The origin of those results can be traced back to the pioneering work of SCHÜTZENBERGER in the early 60's, who proved that every wsa over a field is equivalent to a minimal wsa that is simulated by every *trim* equivalent wsa [4]. Relational simulations of wsa are also studied in [9], where they are used to reduce the size of wsa. The relationship between functional simulations and the MILNER-PARK notion of bisimulation [26, 27] is discussed in [5, 9].

In this contribution, we investigate simulations for weighted (finite) tree automata (wta). SCHÜTZENBERGER's minimization method was extended to wta over fields in [8, 7]. In addition, relational and functional simulations for wta are probably first used in [12, 13, 19]. Moreover, simulations can be generalized to presentations in algebraic theories [5], which seems to cover all mentioned instances. Here, we extend the results of [2, 3] to wta. In particular, we show that two wta over a ring, NOETHERIAN semiring, or the semiring  $\mathbb{N}$  are equivalent if and only if they are connected by a finite chain of simulations. Moreover, we discuss when the simulations can be replaced by functional and dual functional simulations, which are efficiently computable [19]. Such results are important because they immediately yield decidability of equivalence provided that the semiring is finitely and effectively presented.

## 2 Preliminaries

The set of nonnegative integers is  $\mathbb{N}$ . For every  $k \in \mathbb{N}$ , the set  $\{i \in \mathbb{N} \mid 1 \leq i \leq k\}$  is simply denoted by  $[k]$ . We write  $|A|$  for the cardinality of the set  $A$ . A *semiring* is an algebraic structure  $\mathcal{A} = (A, +, \cdot, 0, 1)$  such that  $(A, +, 0)$  and  $(A, \cdot, 1)$  are monoids, of which the former is commutative, and  $\cdot$  distributes both-sided over finite sums (i.e.,  $a \cdot 0 = 0 = 0 \cdot a$  for every  $a \in A$  and  $a \cdot (b + c) = ab + ac$  and  $(b + c) \cdot a = ba + ca$  for every  $a, b, c \in A$ ). The semiring  $\mathcal{A}$  is *commutative* if  $(A, \cdot, 1)$  is commutative. It is a *ring* if for every  $a \in A$  there exists an *additive inverse*  $-a \in A$  such that  $a + (-a) = 0$ . The set  $U$  is the set  $\{a \in A \mid \exists b \in A: ab = 1 = ba\}$  of (*multiplicative*) *units*. The semiring  $\mathcal{A}$  is a *semifield* if  $U = A \setminus \{0\}$ ; i.e., for every  $a \in A$  there exists a *multiplicative inverse*  $a^{-1} \in A$  such that  $aa^{-1} = 1 = a^{-1}a$ . A *field* is a semifield that is also a ring. For every  $B \subseteq A$  let  $\langle B \rangle_+ = \{b_1 + \dots + b_n \mid n \in \mathbb{N}, b_1, \dots, b_n \in B\}$ . If  $A = \langle B \rangle_+$ , then  $\mathcal{A}$  is *additively generated by B*. Finally, it is *equisubtractive* if for every  $a_1, a_2, b_1, b_2 \in A$  with  $a_1 + b_1 = a_2 + b_2$  there exist  $c_1, c_2, d_1, d_2 \in A$  such that (i)  $a_1 = c_1 + d_1$ , (ii)  $b_1 = c_2 + d_2$ , (iii)  $a_2 = c_1 + c_2$ , and (iv)  $b_2 = d_1 + d_2$ .

The semiring  $\mathcal{A}$  is *zero-sum free* if  $a + b = 0$  implies  $0 \in \{a, b\}$  for every  $a, b \in A$ . Clearly, any nontrivial (i.e.,  $0 \neq 1$ ) ring is not zero-sum free. Moreover,  $\mathcal{A}$  is *zero-divisor free* if  $a \cdot b = 0$  implies  $a = 0 = b$  for every  $a, b \in A$ . All semifields are trivially zero-divisor free. Finally, the semiring  $\mathcal{A}$  is *positive* if it is zero-sum free and zero-divisor free. An infinitary sum operation  $\sum$  is a family  $(\sum_I)_I$

such that  $\sum_I: A^I \rightarrow A$ . We generally write  $\sum_{i \in I} a_i$  instead of  $\sum_I (a_i)_{i \in I}$ . The semiring  $\mathcal{A}$  together with the infinitary sum operation  $\sum$  is *complete* [11, 18, 17, 20] if

- $\sum_{i \in \{j_1, j_2\}} a_i = a_{j_1} + a_{j_2}$  for all  $j_1 \neq j_2$  and  $a_{j_1}, a_{j_2} \in A$ ,
- $\sum_{i \in I} a_i = \sum_{j \in J} (\sum_{i \in I_j} a_i)$  for every index set  $I$ , partition  $(I_j)_{j \in J}$  of  $I$ , and  $(a_i)_{i \in I} \in A^I$ , and
- $a \cdot (\sum_{i \in I} a_i) = \sum_{i \in I} aa_i$  and  $(\sum_{i \in I} a_i) \cdot a = \sum_{i \in I} a_i a$  for every  $a \in A$ , index set  $I$ , and  $(a_i)_{i \in I} \in A^I$ .

An  $\mathcal{A}$ -semimodule is a commutative monoid  $(B, +, 0)$  together with an action  $\cdot: A \times B \rightarrow B$ , written as juxtaposition, such that for every  $a, a' \in A$  and  $b, b' \in B$

- $(a + a')b = ab + a'b$  and  $a(b + b') = ab + ab'$ ,
- $0b = 0 = a0$ ,  $1b = b$  and  $(a \cdot a')b = a(a'b)$ .

The semiring  $\mathcal{A}$  is NOETHERIAN if all subsemimodules of every finitely-generated  $\mathcal{A}$ -semimodule are again finitely-generated.

In the following, we often identify index sets of the same cardinality. Let  $X \in A^{I_1 \times J_1}$  and  $Y \in A^{I_2 \times J_2}$  for some finite sets  $I_1, I_2, J_1, J_2$ . We use upper-case letters (like  $C, D, E, X, Y$ ) for matrices and the corresponding lower-case letters for their entries. A matrix  $X \in A^{I \times J}$  is *relational* if  $x_{ij} \in \{0, 1\}$  for every  $i \in I$  and  $j \in J$ . Clearly, a relational matrix defines a relation  $\rho_X \subseteq I \times J$  by  $(i, j) \in \rho_X$  if and only if  $x_{ij} = 1$  (and vice versa). Moreover, we call a relational matrix *functional*, *surjective*, or *injective* if its associated relation has this property. As usual, we denote the *transpose* of a matrix  $X$  by  $X^T$ , and we call  $X$  *nondegenerate* if it has no rows or columns of entirely zeroes. A *diagonal* matrix  $X$  is such that  $x_{ij} = 0$  for every  $i \neq j$ . Finally, the matrix  $X$  is invertible if there exists a matrix  $Y$  such that  $XY = I = YX$  where  $I$  is the unit matrix. The KRONECKER product  $X \otimes Y \in A^{(I_1 \times I_2) \times (J_1 \times J_2)}$  is such that  $(X \otimes Y)_{(i_1, i_2), (j_1, j_2)} = x_{i_1, j_1} y_{i_2, j_2}$  for every  $i_1 \in I_1, i_2 \in I_2, j_1 \in J_1$ , and  $j_2 \in J_2$ . Clearly, the KRONECKER product is, in general, not commutative and  $(1) \in A^{[1]}$  acts both-sided as neutral element. We let  $X^{0, \otimes} = (1)$  and  $X^{i+1, \otimes} = X^{i, \otimes} \otimes X$  for every  $i \in \mathbb{N}$ .

Finally, let us move to trees. A *ranked alphabet* is a finite set  $\Sigma$  together with a mapping  $\text{rk}: \Sigma \rightarrow \mathbb{N}$ . We often just write  $\Sigma$  for a ranked alphabet and assume that the mapping  $\text{rk}$  is implicit. We write  $\Sigma_k = \{\sigma \in \Sigma \mid \text{rk}(\sigma) = k\}$  for the set of all  $k$ -ary symbols. The set of  $\Sigma$ -trees is the smallest set  $T_\Sigma$  such that  $\sigma(t_1, \dots, t_k) \in T_\Sigma$  for all  $\sigma \in \Sigma_k$  and  $t_1, \dots, t_k \in T_\Sigma$ . A *tree series* is a mapping  $\varphi: T_\Sigma \rightarrow A$ . The set of all such tree series is denoted by  $A\langle\langle T_\Sigma \rangle\rangle$ . For every  $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$  and  $t \in T_\Sigma$ , we often write  $(\varphi, t)$  instead of  $\varphi(t)$ . Let  $\square$  be a distinguished nullary symbol such that  $\square \notin \Sigma$ . A  $\Sigma$ -context  $c$  is a tree of  $T_{\Sigma \cup \{\square\}}$ , in which the symbol  $\square$  occurs exactly once. The set of all  $\Sigma$ -contexts is denoted by  $C_\Sigma$ . For every  $c \in C_\Sigma$  and  $t \in T_\Sigma$ , we write  $c[t]$  for the  $\Sigma$ -tree obtained by replacing the unique occurrence of  $\square$  in  $c$  by  $t$ .

A *weighted tree automaton (over  $\mathcal{A}$ )*, for short: wta, is a system  $(\Sigma, Q, \mu, F)$  with

- an input ranked alphabet  $\Sigma$ ,
- a finite set  $Q$  of *states*,
- transitions  $\mu = (\mu_k)_{k \in \mathbb{N}}$  such that  $\mu_k: \Sigma_k \rightarrow A^{Q^k \times Q}$  for every  $k \in \mathbb{N}$ , and
- a *final weight* vector  $F \in A^Q$ .

Next, let us introduce the semantics  $\|M\|$  of  $M$ . We first define the function  $h_\mu: T_\Sigma \rightarrow A^Q$  for every  $\sigma \in \Sigma_k$  and  $t_1, \dots, t_k \in T_\Sigma$  by

$$h_\mu(\sigma(t_1, \dots, t_k)) = (h_\mu(t_1) \otimes \dots \otimes h_\mu(t_k)) \cdot \mu_k(\sigma) ,$$

where the final product  $\cdot$  is the classical matrix product. Then  $(\|M\|, t) = h_\mu(t)F$  for every  $t \in T_\Sigma$ , where the product is the usual inner (dot) product.

Let  $f: A \rightarrow \{0, 1\}$  be such that  $f(0) = 0$  and  $f(a) = 1$  for all  $a \in A \setminus \{0\}$ . The Boolean wta  $f(M)$  (i.e., essentially an unweighted tree automaton) corresponding to  $M$  is  $(\Sigma, Q, \mu', F')$  where

- $\mu'_k(\sigma)_{w,q} = f(\mu_k(\sigma)_{w,q})$  for every  $\sigma \in \Sigma_k$ ,  $w \in Q^k$ , and  $q \in Q$ , and
- $F'(q) = f(F(q))$  for every  $q \in Q$ .

The wta  $M$  is *trim* if every state is accessible and co-accessible in  $f(M)$ . In other words, the wta  $M$  is trim if  $f(M)$  is trim.

### 3 Simulation

Simulations of automata were defined in [5, 15] in order to provide a structural characterization of equivalent automata. We will essentially follow the presentation of [2] here.

**Definition 1.** Let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$  be wta. Then  $M$  simulates  $N$  if there exists a matrix  $X \in A^{Q \times P}$  such that

- (i)  $F = XG$ , and
- (ii)  $\mu_k(\sigma)X = X^{k, \otimes} \cdot \nu_k(\sigma)$  for every  $\sigma \in \Sigma_k$ .

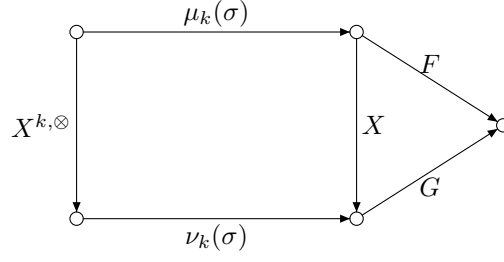
The matrix  $X$  is called *transfer matrix*, and we write  $M \xrightarrow{X} N$  if  $M$  simulates  $N$  with transfer matrix  $X$ .

Note that  $X_{i_1 \dots i_k, j_1 \dots j_k}^{k, \otimes} = \prod_{\ell=1}^k x_{i_\ell, j_\ell}$ . We illustrate Definition 1 in Fig. 1. If  $M \xrightarrow{X} M'$  and  $M' \xrightarrow{Y} N$ , then  $M \xrightarrow{XY} N$ . Thus, simulations define a preorder on wta.

**Theorem 2.** If  $M$  simulates  $N$ , then  $M$  and  $N$  are equivalent.

*Proof.* Let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$ , and let  $X \in A^{Q \times P}$  be a transfer matrix. We claim that  $h_\mu(t)X = h_\nu(t)$  for every  $t \in T_\Sigma$ . We prove this by induction on  $t$ . Let  $t = \sigma(t_1, \dots, t_k)$  for some  $\sigma \in \Sigma_k$  and  $t_1, \dots, t_k \in T_\Sigma$ .

$$\begin{aligned} h_\mu(\sigma(t_1, \dots, t_k))X &= (h_\mu(t_1) \otimes \dots \otimes h_\mu(t_k)) \cdot \mu_k(\sigma)X \\ &= (h_\mu(t_1) \otimes \dots \otimes h_\mu(t_k)) \cdot X^{k, \otimes} \cdot \nu_k(\sigma) = (h_\mu(t_1)X \otimes \dots \otimes h_\mu(t_k)X) \cdot \nu_k(\sigma) \\ &= (h_\nu(t_1) \otimes \dots \otimes h_\nu(t_k)) \cdot \nu_k(\sigma) = h_\nu(\sigma(t_1, \dots, t_k)) \end{aligned}$$



**Fig. 1.** Illustration of simulation.

With this claim, the statement can now be proved easily. For every  $t \in T_\Sigma$

$$(\|M\|, t) = h_\mu(t)F = h_\mu(t)XG = h_\nu(t)G = (\|N\|, t) . \quad \square$$

**Lemma 3.** *Let  $M$  and  $N$  be trim wta such  $M \xrightarrow{X} N$ . If (i)  $X$  is functional or (ii)  $\mathcal{A}$  is positive, then  $X$  is nondegenerate.*

*Proof.* Let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$ . Moreover, let

$$J = \{p \in P \mid \forall q \in Q: x_{qp} = 0\} .$$

Then  $\nu_k(\sigma)_{w,j} = 0$  for every  $\sigma \in \Sigma_k$ ,  $w \in (P \setminus J)^k$ , and  $j \in J$ . This is seen as follows. Since  $\mu_k(\sigma)X = X^{k,\otimes} \cdot \nu_k(\sigma)$  we obtain

$$\sum_{q \in Q} \mu_k(\sigma)_{q_1 \dots q_k, q} \cdot x_{qj} = 0 = \sum_{p_1, \dots, p_k \in P} \left( \prod_{\ell=1}^k x_{q_\ell, p_\ell} \right) \cdot \nu_k(\sigma)_{p_1 \dots p_k, j} \quad (1)$$

for every  $q_1, \dots, q_k \in Q$  and  $j \in J$ . If  $X$  is functional, then

$$\sum_{p_1, \dots, p_k \in P} \left( \prod_{\ell=1}^k x_{q_\ell, p_\ell} \right) \cdot \nu_k(\sigma)_{p_1 \dots p_k, j} = \nu_k(\sigma)_{\rho_X(q_1) \dots \rho_X(q_k), j} = 0 ,$$

which proves the claim. On the other hand, if  $\mathcal{A}$  is positive, then (1) implies that  $\prod_{\ell=1}^k x_{q_\ell, p_\ell} \cdot \nu_k(\sigma)_{p_1 \dots p_k, j} = 0$  for every  $p_1, \dots, p_k \in P$ . Since for every  $p_\ell \notin J$ , there exists  $q_\ell$  such that  $x_{q_\ell, p_\ell} \neq 0$  and  $\prod_{\ell=1}^k x_{q_\ell, p_\ell} \neq 0$  by zero-divisor freeness, we conclude that  $\nu_k(\sigma)_{p_1 \dots p_k, j} = 0$  for every  $p_1, \dots, p_k \in P \setminus J$ , which again proves the claim. Consequently, all states of  $J$  are unreachable. Since  $N$  is trim, we conclude  $J = \emptyset$ , and thus,  $X$  has no column of zeroes.

If  $X$  is functional, then it clearly has no row of zeroes. To prove that  $X$  has no row of zeroes in the remaining case, let  $I = \{q \in Q \mid \forall p \in P: x_{qp} = 0\}$ . Then  $F_i = 0$  and  $\mu_k(\sigma)_{q_1 \dots q_k, q} = 0$  for every  $\sigma \in \Sigma_k$ ,  $q \in Q \setminus I$ ,  $q_1, \dots, q_k \in Q$ , and  $i \in I$  such that  $q_\ell = i$  for some  $\ell \in [k]$ . Clearly,  $F_i = \sum_{p \in P} x_{ip} G_p = 0$  for every  $i \in I$ . Moreover, since  $\mu_k(\sigma)X = X^{k,\otimes} \cdot \nu_k(\sigma)$  we obtain

$$\sum_{q \in Q} \mu_k(\sigma)_{q_1 \dots q_k, q} \cdot x_{qp} = \sum_{p_1, \dots, p_k \in P} \left( \prod_{\ell=1}^k x_{q_\ell, p_\ell} \right) \cdot \nu_k(\sigma)_{p_1 \dots p_k, p} = 0 \quad (2)$$

for every  $q_1, \dots, q_k \in Q$ ,  $p \in P$ , and  $i \in I$  such that  $q_\ell = i$  for some  $\ell \in [k]$ . Since  $\mathcal{A}$  is positive, (2) implies that  $\mu_k(\sigma)_{q_1 \dots q_k, q} \cdot x_{qp} = 0$  for every  $q \in Q$ . However, for all  $q \in Q \setminus I$ , there exists  $p \in P$  such that  $x_{qp} \neq 0$  because  $q \notin I$ . Consequently,  $\mu_k(\sigma)_{q_1 \dots q_k, q} = 0$  by zero-divisor freeness, which proves the claim. Thus, all states of  $I$  are unreachable. Since  $M$  is trim, we conclude  $I = \emptyset$ , and thus,  $X$  has no row of zeroes.  $\square$

**Definition 4 (see [19, Def. 1]).** Let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$  be wta. A surjective function  $\rho: Q \rightarrow P$  is a forward simulation from  $M$  to  $N$  if

- (i)  $F_q = G_{\rho(q)}$  for every  $q \in Q$ , and
- (ii) for every  $p \in P$ ,  $\sigma \in \Sigma_k$ , and  $q_1, \dots, q_k \in Q$

$$\sum_{q \in Q: \rho(q)=p} \mu_k(\sigma)_{q_1 \dots q_k, q} = \nu_k(\sigma)_{\rho(q_1) \dots \rho(q_k), p} .$$

Finally, we say that  $M$  forward simulates  $N$ , written  $M \rightarrow N$ , if there exists a forward simulation from  $M$  to  $N$ .

**Lemma 5.** Let  $M$  and  $N$  be wta such that  $N$  is trim. Then  $M \rightarrow N$  if and only if there exists a functional transfer matrix  $X$  such that  $M \xrightarrow{X} N$ .

*Proof.* Let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$ . First suppose that  $M \xrightarrow{X} N$  with functional  $X \in A^{Q \times P}$ . Then  $\rho_X: Q \rightarrow P$  is a surjective function by Lemma 3. Conversely, if  $M \rightarrow N$  with the forward simulation  $\rho: Q \rightarrow P$ , then  $\rho$  induces a surjective functional matrix  $X \in A^{Q \times P}$  such that  $\rho_X = \rho$ .

Let  $X \in A^{Q \times P}$  be a surjective, functional matrix. It remains to prove that the conditions that (1)  $X$  is a transfer matrix and (2)  $\rho_X$  is a forward simulation are equivalent. We discuss the two items of Definitions 1 and 4 separately.

- (i)  $F = XG$  if and only if  $F_q = G_{\rho(q)}$  for every  $q \in Q$ .
- (ii) for every  $\sigma \in \Sigma_k$ ,  $q_1, \dots, q_k \in Q$ , and  $p \in P$

$$\begin{aligned} (\mu_k(\sigma)X)_{q_1 \dots q_k, p} &= \sum_{q \in Q: \rho_X(q)=p} \mu_k(\sigma)_{q_1 \dots q_k, q} \\ (X^{k, \otimes} \cdot \nu_k(\sigma))_{q_1 \dots q_k, p} &= \nu_k(\sigma)_{\rho_X(q_1) \dots \rho_X(q_k), p} . \end{aligned}$$

Thus,  $X$  is a transfer matrix if and only if  $\rho_X$  is a forward simulation, which proves the statement.  $\square$

**Definition 6 (see [19, Def. 16]).** Let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$  be wta. A surjective function  $\rho: Q \rightarrow P$  is a backward simulation from  $M$  to  $N$  if

- (i)  $\sum_{q \in Q: \rho(q)=p} F_q = G_p$  for every  $p \in P$ , and
- (ii) for every  $q \in Q$ ,  $\sigma \in \Sigma_k$ , and  $p_1, \dots, p_k \in P$

$$\sum_{\substack{q_1, \dots, q_k \in Q \\ \rho(q_1)=p_1, \dots, \rho(q_k)=p_k}} \mu_k(\sigma)_{q_1 \dots q_k, q} = \nu_k(\sigma)_{p_1 \dots p_k, \rho(q)} .$$

Finally, we say that  $M$  backward simulates  $N$ , written  $M \leftarrow N$ , if there exists a backward simulation from  $M$  to  $N$ .

**Lemma 7.** *Let  $M$  and  $N$  be wta such that  $N$  is trim. Then  $M \leftarrow N$  if and only if there exists a transfer matrix  $X$  such that  $X^T$  is functional and  $N \xrightarrow{X} M$ .*

*Proof.* Let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$ . First, suppose that  $N \xrightarrow{X} M$  with the transfer matrix  $X \in A^{P \times Q}$  such that  $X^T$  is functional. Let  $Y = X^T$ . Then  $\rho_Y: Q \rightarrow P$  is a surjective function by Lemma 3. Conversely, if  $M \leftarrow N$  with the backward simulation  $\rho: Q \rightarrow P$ , then  $\rho$  again induces a surjective, functional matrix  $X \in A^{Q \times P}$  such that  $\rho_X = \rho$ .

Let  $X \in A^{Q \times P}$  be a surjective, functional matrix. It remains to prove that the conditions that (1)  $X^T$  is a transfer matrix and (2)  $\rho_X$  is a backward simulation are equivalent. We discuss the two items of Definitions 1 and 6 separately.

- (i)  $G = X^T F$  if and only if  $G_p = \sum_{q \in Q: \rho_X(q)=p} F_q$  for every  $p \in P$ .
- (ii) for every  $\sigma \in \Sigma_k$ ,  $p_1, \dots, p_k \in P$ , and  $q \in Q$

$$\begin{aligned} (\nu_k(\sigma) X^T)_{p_1 \dots p_k, q} &= \nu_k(\sigma)_{p_1 \dots p_k, \rho_X(q)} \\ ((X^T)^{k, \otimes} \cdot \mu_k(\sigma))_{p_1 \dots p_k, q} &= \sum_{\substack{q_1, \dots, q_k \in Q \\ \rho_X(q_1)=p_1, \dots, \rho_X(q_k)=p_k}} \mu_k(\sigma)_{q_1 \dots q_k, q} . \end{aligned}$$

Thus,  $X^T$  is a transfer matrix if and only if  $\rho_X$  is a backward simulation, which proves the statement.  $\square$

**Lemma 8.** *If  $A = \langle U \rangle_+$ , then for every  $X \in A^{Q \times P}$  there exist matrices  $C, E, D$  such that*

- $X = CED$ ,
- $C^T$  and  $D$  are functional, and
- $E$  is an invertible diagonal matrix.

*If (i)  $X$  is nondegenerate or (ii)  $\mathcal{A}$  has (nontrivial) zero-sums, then  $C^T$  and  $D$  can be chosen to be surjective.*

*Proof.* For every  $q \in Q$  and  $p \in P$ , let  $\ell_{qp} \in \mathbb{N}$  and  $u_{qp1}, \dots, u_{qp\ell_{qp}} \in U$  be such that  $x_{qp} = \sum_{i=1}^{\ell_{qp}} u_{qpi}$ . In addition, let

$$J = \{(q, i, p) \mid q \in Q, p \in P, i \in [\ell_{qp}]\} .$$

Finally, let  $\pi_1: J \rightarrow Q$  and  $\pi_3: J \rightarrow P$  be such that  $\pi_1(\langle q, i, p \rangle) = q$  and  $\pi_3(\langle q, i, p \rangle) = p$  for every  $\langle q, i, p \rangle \in J$ . Then we set  $C^T$  and  $D$  to the functional matrices represented by  $\pi_1$  and  $\pi_3$ , respectively. Together with the diagonal matrix  $E$  such that  $e_{\langle q, i, p \rangle, \langle q, i, p \rangle} = u_{qpi}$  for every  $\langle q, i, p \rangle \in J$ , we obtain  $X = CED$ . For every  $q \in Q$  and  $p \in P$  we have

$$\sum_{j_1, j_2 \in J} c_{q, j_1} e_{j_1, j_2} d_{j_2, p} = \sum_{i=1}^{\ell_{qp}} e_{\langle q, i, p \rangle, \langle q, i, p \rangle} = \sum_{i=1}^{\ell_{qp}} u_{qpi} = x_{qp} .$$

It is clear that  $C^T$  and  $D$  are functional matrices. Moreover,  $E$  is an invertible diagonal matrix because  $EE^{-1} = I = E^{-1}E$  where  $E^{-1}$  is the matrix obtained from  $E$  by inverting each nonzero element. If  $X$  is nondegenerate, then  $C^T$  and  $D$  are surjective. Finally, if there are zero-sums, then for every  $q \in Q$  and  $p \in P$  there exist  $u, v \in U$  such that  $x_{qp} = 0 = u + v$ , which yields that we can choose  $\ell_{qp} > 0$ . This completes the proof.  $\square$

**Lemma 9.** *Let  $\mathcal{A}$  be equisubtractive. Moreover, let  $R \in A^Q$  and  $C \in A^P$  be such that  $\sum_{q \in Q} r_q = \sum_{p \in P} c_p$ . Then there exists a matrix  $X \in A^{Q \times P}$  with row sums  $R$  and column sums  $C$ ; i.e.,  $\sum_{q \in Q} x_{qp} = c_p$  for every  $p \in P$  and  $\sum_{p \in P} x_{qp} = r_q$  for every  $q \in Q$ .*

*Proof.* If  $|Q| \leq 1$  or  $|P| \leq 1$ , then the statement is trivially true. Otherwise, select  $i \in Q$  and  $j \in P$ , and let  $Q' = Q \setminus \{i\}$  and  $P' = P \setminus \{j\}$ . By assumption

$$\sum_{q \in Q'} r_q + r_i = \sum_{p \in P'} c_p + c_j .$$

Thus, by equisubtractivity there exist  $a, c'_j, r'_i, x_{ij} \in A$  such that

$$\sum_{q \in Q'} r_q = a + c'_j \quad r_i = r'_i + x_{ij} \quad \sum_{p \in P'} c_p = a + r'_i \quad c_j = c'_j + x_{ij} .$$

Continuing the row decomposition, we obtain  $Y \in A^{Q'}$  and  $R' \in A^{Q'}$  such that  $r_q = r'_q + y_q$  for every  $q \in Q'$  and  $\sum_{q \in Q'} r'_q = a$ . In a similar manner we perform column decomposition to obtain  $Y' \in A^{P'}$  and  $C' \in A^{P'}$  such that  $c_p = c'_p + y'_p$  for every  $p \in P'$  and  $\sum_{p \in P'} c'_p = a$ . Thus, by the induction hypothesis, there exists a matrix  $X' \in A^{Q' \times P'}$  with row sums  $R'$  and column sums  $C'$  because  $\sum_{q \in Q'} r'_q = \sum_{p \in P'} c'_p$ . Then the matrix

$$X = \begin{pmatrix} X' & Y \\ (Y')^T & x_{ij} \end{pmatrix}$$

obviously has the required row and column sums  $R$  and  $C$ , respectively.  $\square$

**Lemma 10.** *If  $X \in A^{Q \times P}$  is functional (respectively, invertible diagonal), then  $X^{k, \otimes}$  is functional (respectively, invertible diagonal) for every  $k \in \mathbb{N}$ .*

*Proof.* Trivial.  $\square$

**Theorem 11.** *Let  $M$  and  $N$  be wta and  $\mathcal{A}$  be equisubtractive with  $A = \langle U \rangle_+$ . Then  $M \xrightarrow{X} N$  if and only if there exist two wta  $M'$  and  $N'$  such that*

- $M \xrightarrow{C} M'$  where  $C^T$  is functional,
- $M' \xrightarrow{E} N'$  where  $E$  is an invertible diagonal matrix, and



–  $N' \xrightarrow{D} N$  where  $D$  is functional.

If  $M$  and  $N$  are trim, then  $M' \leftarrow M$  and  $N' \rightarrow N$ .

*Proof.* Clearly,  $M \xrightarrow{C} M' \xrightarrow{E} N' \xrightarrow{D} N$ , which proves that  $M \xrightarrow{CED} N$ . For the converse, let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$ . Lemma 8 shows that there exist matrices  $C, E, D$  such that

- $X = CED$ ,
- $C^T$  and  $D$  are functional matrices, and
- $E \in A^{I \times I}$  is an invertible diagonal matrix.

Finally, let  $\varphi: I \rightarrow Q$  and  $\psi: I \rightarrow P$  be the functions associated to  $C^T$  and  $D$ . It remains to determine the wta  $M'$  and  $N'$ . We construct  $M' = (\Sigma, I, \mu', F')$  and  $N' = (\Sigma, I, \nu', G')$  with

- $G' = DG$  and
- $F' = EDG$ .

Then  $CF' = CEDG = XG = F$ . Thus, it remains to specify  $\mu'_k(\sigma)$  and  $\nu'_k(\sigma)$  for every  $\sigma \in \Sigma_k$ . To this end, we determine a matrix  $Y \in A^{I^k \times I}$  such that

$$C^{k, \otimes} \cdot Y = \mu_k(\sigma)CE \quad (3)$$

$$YD = E^{k, \otimes} \cdot D^{k, \otimes} \cdot \nu_k(\sigma) \quad (4)$$

Given such a matrix  $Y$ , we then let  $\mu'_k(\sigma) = YE^{-1}$  and  $\nu'_k(\sigma) = (E^{k, \otimes})^{-1} \cdot Y$ . Then

$$\mu_k(\sigma)C = C^{k, \otimes} \cdot \mu'_k(\sigma) \quad \mu'_k(\sigma)E = E^{k, \otimes} \cdot \nu'_k(\sigma) \quad \nu'_k(\sigma)D = D^{k, \otimes} \cdot \nu_k(\sigma) \quad .$$

These equalities are displayed in Fig. 2.

Finally, we need to specify the matrix  $Y$ . For every  $q \in Q$  and  $p \in P$ , let  $I_q = \varphi^{-1}(q)$  and  $J_p = \psi^{-1}(p)$ . Obviously,  $Y$  can be decomposed into disjoint (not necessarily contiguous) submatrices  $Y_{q_1 \dots q_k, p} \in A^{(I_{q_1} \times \dots \times I_{q_k}) \times J_p}$  with  $q_1, \dots, q_k \in Q$  and  $p \in P$ . Then (3) and (4) hold if and only if for every  $q_1, \dots, q_k \in Q$  and  $p \in P$  the following two conditions hold:

1. For every  $i \in I$  such that  $\psi(i) = p$ , the sum of the  $i$ -column of  $Y_{q_1 \dots q_k, p}$  is  $\mu_k(\sigma)_{q_1 \dots q_k, \varphi(i)} \cdot e_{i, i}$ .
2. For all  $i_1, \dots, i_k \in I$  such that  $\varphi(i_j) = q_j$  for every  $j \in [k]$ , the sum of the  $(i_1, \dots, i_k)$ -row of  $Y_{q_1 \dots q_k, p}$  is  $\prod_{j=1}^k e_{i_j, i_j} \cdot \nu_k(\sigma)_{\psi(i_1) \dots \psi(i_k), p}$ .

Those two conditions are compatible because

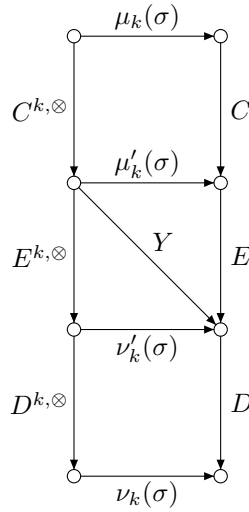
$$\begin{aligned} & \sum_{\substack{i \in I \\ \psi(i) = p}} \mu_k(\sigma)_{q_1 \dots q_k, \varphi(i)} \cdot e_{i, i} = (\mu_k(\sigma)CED)_{q_1 \dots q_k, p} = (\mu_k(\sigma)X)_{q_1 \dots q_k, p} \\ & \stackrel{\dagger}{=} (X^{k, \otimes} \cdot \nu_k(\sigma))_{q_1 \dots q_k, p} = (C^{k, \otimes} \cdot E^{k, \otimes} \cdot D^{k, \otimes} \cdot \nu_k(\sigma))_{q_1 \dots q_k, p} \\ & = \sum_{\substack{i_1, \dots, i_k \in I \\ \forall j \in [k]: \varphi(i_j) = q_j}} \left( \prod_{j=1}^k e_{i_j, i_j} \right) \cdot \nu_k(\sigma)_{\psi(i_1) \dots \psi(i_k), p} \quad . \end{aligned}$$

Consequently, the row and column sums of the submatrices  $Y_{q_1 \dots q_k, p}$  are consistent, which yields that we can determine all the submatrices (and thus the whole matrix) by Lemma 9.

If  $M$  and  $N$  are trim, then either

- (a)  $\mathcal{A}$  is zero-sum free (and thus positive because it is additively generated by its units), in which case  $X$  is nondegenerate by Lemma 3, or
- (b)  $\mathcal{A}$  has nontrivial zero-sums.

In both cases, Lemma 8 shows that the matrices  $C^T$  and  $D$  are surjective, which yields the additional statement by Lemmata 5 and 7.  $\square$



**Fig. 2.** Illustration of the relations between the matrices in the proof of Theorem 11.

## 4 Category of simulations

In this section our aim is to show that several well-known constructions of wta are *functorial*: they may be extended to simulations in a functorial way. Below we will only deal with the sum, HADAMARD product,  $\sigma_0$ -product, and  $\sigma_0$ -iteration (cf. [14]). Scalar OI-substitution,  $^\dagger$  [6], homomorphism, quotient, and top-concatenation [14] may be covered in a similar fashion.

Throughout this section, let  $\mathcal{A}$  be commutative. Let  $M = (\Sigma, Q, \mu, F)$ ,  $M' = (\Sigma, Q', \mu', F')$ , and  $M'' = (\Sigma, Q'', \mu'', F'')$  be wta. We already remarked that, if  $M \xrightarrow{X} M'$  and  $M' \xrightarrow{Y} M''$ , then  $M \xrightarrow{XY} M''$ . Moreover,  $M \xrightarrow{I} M$  with the unit matrix  $I \in A^{Q \times Q}$ . Thus, wta over the alphabet  $\Sigma$  form a category  $\mathbf{Sim}_\Sigma$ .

In the following, let  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$  be wta such that  $Q \cap P = \emptyset$ .

**Definition 12.** The sum  $M + N$  of  $M$  and  $N$  is the wta  $(\Sigma, Q \cup P, \kappa, H)$  where  $H = \langle F, G \rangle = \begin{pmatrix} F \\ G \end{pmatrix}$  and

$$\kappa_k(\sigma)_{q_1 \dots q_k, q} = (\mu_k(\sigma) + \nu_k(\sigma))_{q_1 \dots q_k, q} = \begin{cases} \mu_k(\sigma)_{q_1 \dots q_k, q} & \text{if } q, q_1, \dots, q_k \in Q \\ \nu_k(\sigma)_{q_1 \dots q_k, q} & \text{if } q, q_1, \dots, q_k \in P \\ 0 & \text{otherwise.} \end{cases}$$

for all  $\sigma \in \Sigma_k$  and  $q, q_1, \dots, q_k \in Q \cup P$ .

It is well-known that  $\|M + N\| = \|M\| + \|N\|$ . Next, we extend the sum construction to simulations. To this end, let  $M \xrightarrow{X} M'$  with  $M' = (\Sigma, Q', \mu', F')$ , and let  $N \xrightarrow{Y} N'$  with  $N' = (\Sigma, P', \nu', G')$ .

**Definition 13.** The sum  $X + Y \in A^{(Q \cup P) \times (Q' \cup P')}$  of the transfer matrices  $X$  and  $Y$  is

$$X + Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

**Proposition 14.** We have  $(M + N) \xrightarrow{X+Y} (M' + N')$ .

*Proof.* We only need to verify the two conditions of Definition 1. For every  $\sigma \in \Sigma_k$  we have

$$\begin{aligned} (\mu_k(\sigma) + \nu_k(\sigma)) \cdot (X + Y) &= \mu_k(\sigma)X + \nu_k(\sigma)Y \\ &= X^{k, \otimes} \cdot \mu'_k(\sigma) + Y^{k, \otimes} \cdot \mu'_k(\sigma) = (X + Y)^{k, \otimes} \cdot (\mu'_k(\sigma) + \nu'_k(\sigma)) \end{aligned}$$

and  $\langle F, G \rangle = \langle XF', YG' \rangle = (X + Y) \cdot \langle F', G' \rangle$ , which completes the proof.  $\square$

**Proposition 15.** The function  $+$ , which is defined on wta and transfer matrices, is a functor  $\mathbf{Sim}_{\Sigma}^2 \rightarrow \mathbf{Sim}_{\Sigma}$ .

*Proof.* It is a routine matter to verify that identity transfer matrices are preserved and  $(X + Y) \cdot (X' + Y') = XX' + YY'$  for all composable transfer matrices  $X, X', Y, Y'$ .  $\square$

**Definition 16.** Let  $\sigma_0$  be a distinguished symbol in  $\Sigma_0$ . The  $\sigma_0$ -product  $M \cdot_{\sigma_0} N$  of  $M$  with  $N$  is the wta  $(\Sigma, Q \cup P, \kappa, H)$  such that

$$H = \langle F, 0 \rangle = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

and for each  $\sigma \in \Sigma_k$  with  $\sigma \neq \sigma_0$ ,

$$\kappa_k(\sigma)_{q_1 \dots q_k, q} = \begin{cases} \mu_k(\sigma)_{q_1 \dots q_k, q} & \text{if } q, q_1, \dots, q_k \in Q \\ \mu_0(\sigma_0)_q \cdot \sum_{p \in P} \nu_k(\sigma)_{q_1 \dots q_k, p} G_p & \text{if } q \in Q \text{ and } q_1, \dots, q_k \in P \\ \nu_k(\sigma)_{q_1 \dots q_k, q} & \text{if } q, q_1, \dots, q_k \in P \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\kappa_0(\sigma_0)_q = \begin{cases} \mu_0(\sigma_0)_q \cdot \sum_{p \in P} \nu_0(\sigma_0)_p G_p & \text{if } q \in Q \\ \nu_0(\sigma_0)_q & \text{if } q \in P. \end{cases}$$

It is known that  $\|M \cdot_{\sigma_0} N\| = \|M\| \cdot_{\sigma_0} \|N\|$ . We extend this construction to simulations. To this end, let  $M \xrightarrow{X} M'$  and  $N \xrightarrow{Y} N'$ . Then we define  $X \cdot_{\sigma_0} Y = X + Y$ . The next proposition can be verified by a routine calculation.

**Proposition 17.** *The function  $\cdot_{\sigma_0}$ , which is defined on wta and transfer matrices, is a functor  $\mathbf{Sim}_{\Sigma}^2 \rightarrow \mathbf{Sim}_{\Sigma}$ .*

**Definition 18.** *The HADAMARD product  $M \cdot_H N$  is the wta  $(\Sigma, Q \times P, \kappa, H)$  where  $H = F \otimes G$  and  $\kappa_k(\sigma) = \mu_k(\sigma) \otimes \nu_k(\sigma)$  for all  $\sigma \in \Sigma_k$ .*

We again extend the construction to simulations. If  $M \xrightarrow{X} M'$  and  $N \xrightarrow{Y} N'$ , then we define  $X \cdot_H Y = X \otimes Y$ .

**Proposition 19.** *The function  $\cdot_H$ , which is defined on wta and transfer matrices, is a functor  $\mathbf{Sim}_{\Sigma}^2 \rightarrow \mathbf{Sim}_{\Sigma}$ .*

Finally, we deal with iteration. Let  $\sigma_0$  be a fixed symbol in  $\Sigma_0$ . Here we assume that  $\mathcal{A}$  is complete. Thus,  $\mathcal{A}$  comes with a star operation  $a^* = \sum_{n \in \mathbb{N}} a^n$  for every  $a \in A$ .

**Definition 20.** *The  $\sigma_0$ -iteration  $M^{*\sigma_0}$  of  $M$  is the wta  $(\Sigma, Q, \kappa, F)$  where*

$$\kappa_k(\sigma)_{q_1 \dots q_k, q} = \mu_k(\sigma)_{q_1 \dots q_k, q} + \|M\|(\sigma_0)^* \cdot \sum_{p \in Q} \mu_k(\sigma)_{q_1 \dots q_k, p} F_p$$

for all  $\sigma \in \Sigma_k \setminus \{\sigma_0\}$  and  $\kappa_0(\sigma_0) = \mu_0(\sigma_0)$ .

If  $M \xrightarrow{X} M'$ , then we define  $X^{*\sigma_0} = X$ .

**Proposition 21.** *The  $\sigma_0$ -iteration, which is defined on wta and transfer matrices, is a functor  $\mathbf{Sim}_{\Sigma} \rightarrow \mathbf{Sim}_{\Sigma}$ .*

*Remark 22.* Several subcategories of  $\mathbf{Sim}_{\Sigma}$  are also of interest, for example the categories formed by the relational or functional simulations and their duals. The above constructions are preserved by these special kinds of simulations.

## 5 Joint reduction

Next we will establish equivalence results using the approach called *joint reduction* in [3]. Let  $V \subseteq A^I$  be a set of vectors for a finite set  $I$ . Then the  $\mathcal{A}$ -semimodule generated by  $V$  is denoted by  $\langle V \rangle$ . Given two wta  $M = (\Sigma, Q, \mu, F)$  and  $N = (\Sigma, P, \nu, G)$  with  $Q \cap P = \emptyset$ , we first compute  $M + N = (\Sigma, Q \cup P, \mu', F')$  as defined in Section 4. Now the aim is to compute a finite set  $V \subseteq A^{Q \cup P}$  such that

- (i)  $(v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma) \in \langle V \rangle$  for every  $\sigma \in \Sigma_k$  and  $v_1, \dots, v_k \in V$ , and
- (ii)  $v_1 F = v_2 G$  for every  $(v_1, v_2) \in V$  such that  $v_1 \in A^Q$  and  $v_2 \in A^P$ .

With such a finite set  $V$  we can now construct a wta  $M' = (\Sigma, V, \nu', G')$  with  $G'_v = vF'$  for every  $v \in V$  and

$$\sum_{v \in V} \nu'_k(\sigma)_{v_1 \dots v_k, v} \cdot v = (v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma)$$

for every  $\sigma \in \Sigma_k$  and  $v_1, \dots, v_k \in V$ . It remains to prove that  $M'$  simulates  $M + N$ . To this end, let  $X = (v)_{v \in V}$ , where each  $v \in V$  is a row vector. Then for every  $\sigma \in \Sigma_k$ ,  $v_1, \dots, v_k \in V$ , and  $q \in Q \cup P$ , we have

$$\begin{aligned} (\nu'_k(\sigma)X)_{v_1 \dots v_k, q} &= \sum_{v \in V} \nu'_k(\sigma)_{v_1 \dots v_k, v} \cdot v_q = \left( \sum_{v \in V} \nu'_k(\sigma)_{v_1 \dots v_k, v} \cdot v \right)_q \\ &= ((v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma))_q = \sum_{q_1, \dots, q_k \in Q \cup P} (v_1)_{q_1} \cdots (v_k)_{q_k} \cdot \mu'_k(\sigma)_{q_1 \dots q_k, q} \\ &= (X^{k, \otimes} \cdot \mu'_k(\sigma))_{v_1 \dots v_k, q} . \end{aligned}$$

Moreover, if we let  $X_1$  and  $X_2$  be the restrictions of  $X$  to the entries of  $Q$  and  $P$ , respectively, then we have  $\nu'_k(\sigma)X_1 = X_1^{k, \otimes} \cdot \mu'_k(\sigma)$  and  $\nu'_k(\sigma)X_2 = X_2^{k, \otimes} \cdot \mu'_k(\sigma)$ . In addition,  $G'_v = vF' = \sum_{q \in Q \cup P} v_q F'_q = (XF')_v$  for every  $v \in V$ , which proves that  $M' \xrightarrow{X} (M + N)$ . Since  $v_1 F = v_2 G$  for every  $(v_1, v_2) \in V$ , we have  $G'_{(v_1, v_2)} = (v_1, v_2)F' = v_1 F + v_2 G = (1 + 1)v_1 F = (1 + 1)v_2 G$ . Now, let  $G''_{(v_1, v_2)} = v_1 F = v_2 G$  for every  $(v_1, v_2) \in V$ . Then

$$\begin{aligned} G''_v &= v_1 F = \sum_{q \in Q} v_q F_q = (X_1 F)_v \\ &= v_2 G = \sum_{p \in P} v_p G_p = (X_2 G)_v \end{aligned}$$

for every  $v = (v_1, v_2) \in V$ . Consequently,  $M'' \xrightarrow{X_1} M$  and  $M'' \xrightarrow{X_2} N$ , where  $M'' = (\Sigma, V, \nu', G'')$ . This proves the next theorem.

**Theorem 23.** *Let  $M$  and  $N$  be two equivalent wta. If there exists a finite set  $V \subseteq A^{Q \cup P}$  with properties (i) and (ii), then there exists a chain of simulations that join  $M$  and  $N$ . In fact, there exists a single wta that simulates both  $M$  and  $N$ .*

## 5.1 Fields

In this section, let  $\mathcal{A}$  be a field. We first recall some notions from [8]. Let  $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$  be a tree series. The *syntactic ideal* of  $\varphi$  is

$$I_\varphi = \{ \psi \in A\langle\langle T_\Sigma \rangle\rangle \mid \sum_{t \in T_\Sigma} (\psi, t)(\varphi, c[t]) = 0 \text{ for all } c \in C_\Sigma \} .$$

Moreover, let  $\equiv$  be the equivalence relation on  $A\langle\langle T_\Sigma \rangle\rangle$  such that  $\psi_1 \equiv \psi_2$  if and only if  $\psi_1 - \psi_2 \in I_\varphi$ . The syntactic algebra is  $[A\langle\langle T_\Sigma \rangle\rangle]_\equiv$ . By [8, Proposition 2] the tree series  $\varphi$  is recognizable if and only if its syntactic algebra has finite dimension. Now, let  $\varphi$  be recognizable, and let  $B$  be a basis of its syntactic algebra. Finally, let  $M_\varphi$  be the obtained canonical weighted tree automaton, which recognizes  $\varphi$ .

**Theorem 24 ([8, p. 453]).** *Every trim wta recognizing  $\varphi$  simulates  $M_\varphi$ .*

Consequently, all equivalent trim wta  $M_1$  and  $M_2$  simulate the canonical wta that recognizes  $\|M\|$ . Using Theorem 11 we can show that there exist wta  $M'_1$ ,  $M'_2$ ,  $N'_1$ , and  $N'_2$  such that

- $M_1 \leftarrow M'_1$ ,
- $M'_1 \xrightarrow{E} N'_1$  with an invertible diagonal matrix  $E$ ,
- $N'_1 \rightarrow M_\varphi$ ,
- $N'_2 \rightarrow M_\varphi$ ,
- $M'_2 \xrightarrow{E'} N'_2$  with an invertible diagonal matrix  $E'$ , and
- $M_2 \leftarrow M'_2$ .

This can be illustrated as follows:

$$M_1 \xleftarrow{\text{backward}} M'_1 \xrightarrow{\text{diagonal}} N'_1 \xrightarrow{\text{forward}} M_\varphi \xleftarrow{\text{forward}} N'_2 \xleftarrow{\text{diagonal}} M'_2 \xrightarrow{\text{backward}} M_2$$

**Theorem 25.** *Every two equivalent trim wta  $M$  and  $N$  over the field  $\mathcal{A}$  can be joined by a chain of simulations. Moreover, there exists a minimal wta  $M_{\|M\|}$  such that  $M$  and  $N$  both simulate  $M_{\|M\|}$ .*

We could have obtained a similar theorem with the help of Theorem 23 because the finite set  $V$  can be obtained as in [7]. The approach in the next section will cover this case.

## 5.2 NOETHERIAN semirings

Now, let  $\mathcal{A}$  be a NOETHERIAN semiring. We construct the finite set  $V$  as follows. Let  $V_0 = \{\mu'_0(\alpha) \mid \alpha \in \Sigma_0\}$  and

$$V_{i+1} = V_i \cup (\{(v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma) \mid \sigma \in \Sigma_k, v_1, \dots, v_k \in V_i\} \setminus \langle V_i \rangle)$$

for every  $i \in \mathbb{N}$ . Then

$$\{0\} \subseteq \langle V_0 \rangle \subseteq \langle V_1 \rangle \subseteq \cdots \subseteq \langle V_k \rangle \subseteq \cdots$$

is stationary after finitely many steps because  $\mathcal{A}$  is NOETHERIAN. Thus, let  $V = V_k$  for some  $k \in \mathbb{N}$  such that  $\langle V_k \rangle = \langle V_{k+1} \rangle$ . Clearly,  $V$  is finite and has property (i). Trivially,  $V \subseteq \{h_\mu(t) \mid t \in T_\Sigma\}$ , so let  $v \in V$  be such that  $v = \sum_{i \in I} (h_\mu(t_i), h_\nu(t_i))$  for some finite index set  $I$  and  $t_i \in T_\Sigma$  for every  $i \in I$ . Then

$$\left( \sum_{i \in I} h_\mu(t_i) \right) F = \sum_{i \in I} (\|M\|, t_i) = \sum_{i \in I} (\|N\|, t_i) = \left( \sum_{i \in I} h_\nu(t_i) \right) G$$

because  $\|M\| = \|N\|$ , which proves property (ii).

**Theorem 26.** *Let  $\mathcal{A}$  be a NOETHERIAN semiring. For every two equivalent wta  $M$  and  $N$  over  $\mathcal{A}$ , there exists a chain of simulations that join  $M$  and  $N$ . In fact, there exists a single wta that simulates both  $M$  and  $N$ .*

*Proof.* Follows from Theorem 23.

Since  $\mathbb{Z}$  forms a NOETHERIAN ring, we obtain the following corollary.

**Corollary 27 (of Theorem 26).** *For every two equivalent wta  $M$  and  $N$  over  $\mathbb{Z}$ , there exists a chain of simulations that join  $M$  and  $N$ . In fact, there exists a single wta that simulates both  $M$  and  $N$ .*

In fact, since  $M+N$  uses only finitely many semiring coefficient, it is sufficient that every finitely generated subsemiring of  $\mathcal{A}$  is contained in a NOETHERIAN subsemiring of  $\mathcal{A}$ . Since every finitely generated commutative ring is NOETHERIAN [24, Cor. IV.2.4 & Prop. X.1.4], we obtain the following corollary.

**Corollary 28 (of Theorem 26).** *For every two equivalent wta  $M$  and  $N$  over the commutative ring  $\mathcal{A}$ , there exists a chain of simulations that join  $M$  and  $N$ . In fact, there exists a single wta that simulates both  $M$  and  $N$ .*

### 5.3 Natural numbers

Finally, let  $\mathcal{A} = \mathbb{N}$  be the semiring of natural numbers. We compute the finite set  $V \subseteq \mathbb{N}^{Q \cup P}$  as follows:

1. Let  $V_0 = \{\mu'_0(\alpha) \mid \alpha \in \Sigma_0\}$  and  $i = 0$ .
2. For every  $v, v' \in V_i$  such that  $v \leq v'$ , replace  $v'$  by  $v' - v$ .
3. Set  $V_{i+1} = V_i \cup (\{(v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma) \mid \sigma \in \Sigma_k, v_1, \dots, v_k \in V_i\} \setminus \langle V_i \rangle)$ .
4. Until  $V_{i+1} = V_i$ , increase  $i$  and repeat step 2.

Clearly, this algorithm terminates since every vector can only be replaced by a smaller vector in step 2 and step 3 only adds a finite number of vectors, which after the reduction in step 2 are pairwise incomparable. Moreover, property (i) trivially holds because at termination  $V_{i+1} = V_i$  after step 3. Consequently, we only need to prove property (ii). To this end, we first prove that  $V \subseteq \langle \{h_{\mu'}(t) \mid t \in T_\Sigma\} \rangle_{+, -}$ . This is trivially true after step 1 because  $\mu'_0(\alpha) = h_{\mu'}(\alpha)$  for every  $\alpha \in \Sigma_0$ . Clearly, the property is preserved in steps 2 and 3. Finally, property (ii) can now be proved as follows. Let  $v \in V$  be such that  $v = \sum_{i \in I_1} (h_\mu(t_i), h_\nu(t_i)) - \sum_{i \in I_2} (h_\mu(t_i), h_\nu(t_i))$  for some finite index sets  $I_1$  and  $I_2$  and  $t_i \in T_\Sigma$  for every  $i \in I_1 \cup I_2$ . Then

$$\begin{aligned} & \left( \sum_{i \in I_1} h_\mu(t_i) - \sum_{i \in I_2} h_\mu(t_i) \right) F = \sum_{i \in I_1} h_\mu(t_i) F - \sum_{i \in I_2} h_\mu(t_i) F \\ &= \sum_{i \in I_1} (\|M\|, t_i) - \sum_{i \in I_2} (\|M\|, t_i) = \sum_{i \in I_1} (\|N\|, t_i) - \sum_{i \in I_2} (\|N\|, t_i) \\ &= \sum_{i \in I_1} h_\nu(t_i) G - \sum_{i \in I_2} h_\nu(t_i) G = \left( \sum_{i \in I_1} h_\nu(t_i) - \sum_{i \in I_2} h_\nu(t_i) \right) G \end{aligned}$$

because  $\|M\| = \|N\|$ .

**Corollary 29 (of Theorem 23).** *For every two equivalent wta  $M$  and  $N$  over  $\mathbb{N}$ , there exists a chain of simulations that join  $M$  and  $N$ . In fact, there exists a single wta that simulates both  $M$  and  $N$ .*

For all finitely and effectively presented semirings, Theorems 25 and 26 and Corollaries 28 and 29, also yield decidability of equivalence for  $M$  and  $N$ . Essentially, we run the trivial semi-decidability test for inequality and a search for the wta that simulates both  $M$  and  $N$  in parallel. We know that either test will eventually return, thus deciding whether  $M$  and  $N$  are equivalent. Conversely, if equivalence is undecidable, then simulation cannot capture equivalence [16].

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